A note on three water cups

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1 Problem Statement

We have three cups of water of sizes $A \geq B \geq C$ respectively. We start with cup A full and cups B and C empty. If we pour water from one cup to another, we can stop only if the source cup becomes empty, or the destination cup becomes full. By repeating this process, we can put this 3-cup system in a number of possible states. I'll show that reaching any such state requires at most $A+1$ pourings.

Disclaimer: this note is a sketch that is meant to describe an approach rather than a formal proof: some parts are not as specific as I'd like to, but I hope that the overall reasoning is correct.

2 Case $A > B + C$

If $A > B + C$, then the cup A will never be empty, and will have at least $A - B - C$ units of water at the bottom. We can just ignore these units, and consider them part of cup A. This reduces this case to $A' = B + C$.

3 Representing states

For the rest of this note, I'll assume $A \leq B + C$.

Let's denote states as tuples (a, b, c) that represent the amount of water in each cup, respectively. We have $a + b + c = A$, so just (b, c) uniquely specifies the state $(A - b - c, b, c)$. Figure 1 depicts the state space using (b, c) as coordinates ($A = 89$, $B = 77$, $C = 62$).

Figure 1 also depicts an example sequence of pourings. We start at state $S_0 = (A, 0, 0)$ in the upper left corner of the diagram, and pour water from

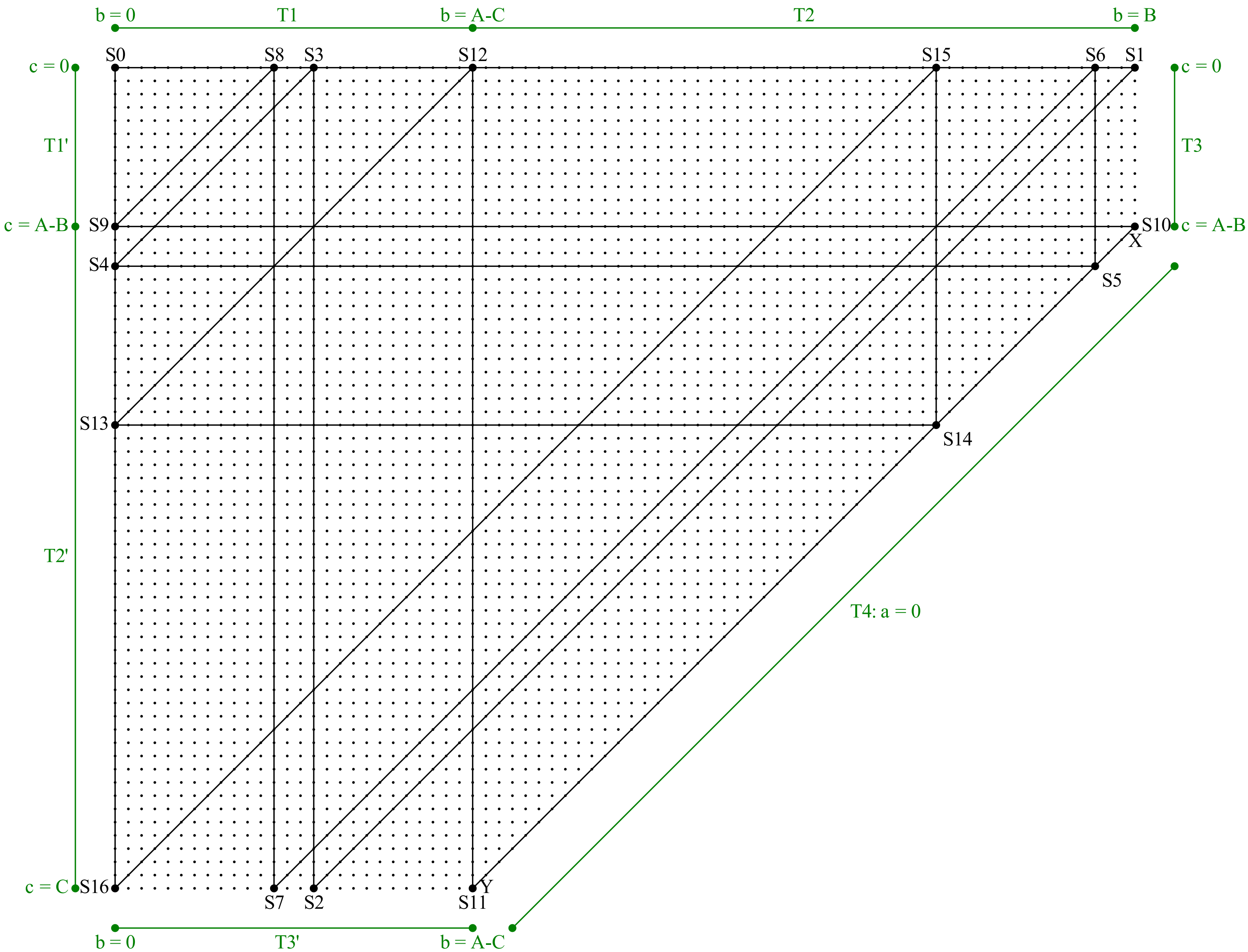

Figure 1: Diagram for the case $A = 89$, $B = 77$, $C = 62$.

A to B, landing at $S_1 = (A - B, B, 0)$ in the upper right corner. Then, we pour water from B to C, obtaining $S_2 = (A - B, B - C, C)$, and so on. In general, let S_k be the k-th state in our path. After 17 moves we're back where we started: $S_{17} = S_0$.

Graphically, pouring water between A and B moves us along a horizontal line, between A and C: along a vertical line, and between B and C: along a diagonal line. In all cases, we stop only once we've reached the boundary of the state diagram: either the source cup has become empty, or the destination cup has become full. By induction, all reachable states lie on the boundary, that is, have at least one cup full or empty.

4 States and moves

As shown in Figure 1, there are 5 *special states*, which have more than one cup full or empty. These are the corners of the diagram: S_0, S_1, S_{10}, S_{11}, and $S_{16} = S_{-1}$. Because the values 10 and 11 depend on our choice of (A, B, C), let's give states S_{10} and S_{11} names that are (A, B, C)-independent: $S_{10} = X$, and $S_{11} = Y$. More precisely, I define:

$$\begin{aligned} S_0 &= (A, 0, 0) \\ S_1 &= (A - B, B, 0) \\ S_{-1} &= (A - C, 0, C) \\ X &= (0, B, A - B) \\ Y &= (0, A - C, C) \end{aligned}$$

Note that if $A = B + C$, then $X = Y$.

States that are not special, I'll call *regular*. Special states are corners, regular states lie on the boundary but are not corners. I'll also distinguish *reversible* moves and *non-reversible* moves. Reversible moves are pouring from a full cup or pouring to an empty cup. Examples of reversible moves: $S_{15} \to S_{14}$ and $S_{10} \to S_9$, examples of non-reversible moves: $S_{15} \to S_0$ and $S_{15} \to S_1$. Reversible moves can be undone, non-reversible moves cannot.

For regular states, reversible moves correspond to moving inside the state space, and not along the boundary. It's easy to see that each regular state has exactly two reversible moves available. Moves from special states are always reversible.

Non-reversible moves always land us at one of the special states. But all of these five special states are easily reachable with regular moves as well: S_0

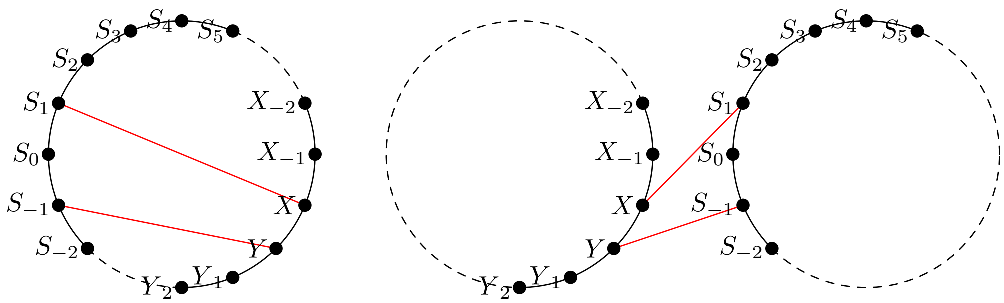

Figure 2: Two possible graphs of reversible moves moves: (a) a single cycle, and (b) two cycles. Regular moves are black, special moves are red.

is the starting state, then we can do $S_0 \to S_1 \to X$, or $S_0 \to S_{-1} \to Y$. For this reason, I'll ignore non-reversible moves altogether from now on.

I'll call the two moves $S_{-1} \leftrightarrow Y$ and $S_1 \leftrightarrow X$ *special*, and all the other (reversible) moves, *regular*. It's easy to see that each state has exactly two regular moves. This means that the sequence of states S_0, S_1, ... is uniquely determined by stipulating that $S_k \to S_{k+1}$ is always a regular move.

Theorem 4.1. *Let m be the total number of states reachable from S_0. The distance between S_0 and any other state s is at most $m/2$.*

Proof. Each node in the graph G of regular moves has exactly two edges, so the graph is a collection of disjoint cycles. States S_{-1}, S_0, and S_1 are connected through regular moves, and so are X and Y, so graph G is either a single cycle or two cycles. See Figure 2.

The distance between any two points on a cycle is at most half the number m of nodes in that cycle. This is because one can go in one of two directions, with distances d_1 and d_2, respectively. Since $d_1+d_2 = m$, we have $\min(d_1, d_2) \leq m/2$.

In the first case, where all states lie on the same cycle in G, even ignoring the special red edges, the assertion clearly holds.

In the second case, assume that the two cycles have m_S and m_X elements, respectively. If s lies on the S cycle, then the $d(S_0, s) \leq m_S/2 \leq m/2$, as required. If s lies on the X cycle, then X itself can be reached from S_0 in two steps ($S_0 \to S_1 \to X$), and s can be reached from X in at most $m_X/2$ steps. So $d(S_0, s) \leq 2 + m_X/2 = (4 + m_x)/2 \leq m/2$. Here, we used the fact that $m_S \geq 4$. This is because states S_{-1}, S_1, and S_2 are all different, except when $B = C$, which I'll analyze separately now.

Case $B = C$ has two cycles: $\{S_0, S_1, S_{-1}\}$ and $\{X, Y, Y', X'\}$, where $X' = (B, 0, A - B)$ and $Y' = (C, A - C, 0)$. The states furthers away from S_0 are

X' and Y', and both can be reached in 3 moves: $S_0 \to S_1 \to X \to X'$ and $S_0 \to S_{-1} \to Y \to Y'$. As a result $m = 7$, and $d(S_0, s) \leq 3 \leq 7/2 = m/2$. $\square$

To use Theorem 4.1, we need a good bound on the number m of reachable states. One such bound is $m \leq A + B + C$, the number of all states at the boundary of the diagram. However, we can often do better: Theorem 4.2 implies a bound $m \leq 2A+3$. This, by Theorem 4.1, gives the bound $d(S_0, s) \leq A + 1$.

Theorem 4.2. *There are at most two reachable regular states with cup A empty. (In Figure 1, these states are S_5 and S_{14}.)*

Proof. The case $A = B + C$ has no such states, so let's assume $A < B + C$.

Let's start with dividing states (a, b, c) into seven types, shown in Figure 1:

$$\begin{aligned}
T_1 &: \quad c = 0 \text{ and } 0 \leq b \leq A - C \\
T_2 &: \quad c = 0 \text{ and } A - C < b < B \\
T_3 &: \quad 0 \leq c \leq A - B \text{ and } b = B \\
T_1' &: \quad b = 0 \text{ and } 0 \leq c \leq A - B \\
T_2' &: \quad b = 0 \text{ and } A - B < c < C \\
T_3' &: \quad 0 \leq b \leq A - C \text{ and } c = C \\
T_4 &: \quad a = 0 \text{ and } b < B \text{ and } c < C
\end{aligned}$$

Each state other than S_0 is of exactly one type. S_0 is both T_1 and T_1', but this should not be a problem.

The assertion can now be restated as: there are at most two reachable T_4 states.

I'll say that two non-T_4 states are *dual* iff the sum of their a's is A, and they lie on the "opposite" sides of the diagram. More precisely:

$$\begin{array}{rcr}
(*, 0, c) \in T_1' & \text{is dual to} & (*, B, A - B - c) \in T_3 \\
(*, 0, c) \in T_2' & \text{is dual to} & (*, A - c, 0) \in T_2 \\
(*, b, C) \in T_3' & \text{is dual to} & (*, A - C - b, 0) \in T_1
\end{array}$$

In addition I define each T_4 state as dual to itself.

Examples of dual pairs: S_9 and S_1, S_3 and S_7, S_{13} and S_{15}.

To prove that there are at most two T_4 states, let s be a such a state, for example, S_5. Let $s = s_0, s_1, s_2, \ldots$ be the sequence of states starting with s, and following regular moves. There are two choices for s_1, we just pick one, the other sequence being s_0, s_{-1}, s_{-2}, ... It can be shown by induction and case analysis, that as long as none of the states s_1, ..., s_k are T_4, then s_k and s_{-k} are dual.

As discussed in Theorem 4.1 and depicted in Figure 2, the graph of reachable states has either one or two cycles of regular moves. If it is a single cycle, then let s be the first S_k of type T_4, and choose the direction of s_k so that $S_0 = s_{-k}$. Since s is the first T_4 state in s_{-k}, ..., $s_0 = s$, by duality, none of the states s_1, ..., s_k are T_4, and $s_k = S_{2k}$ is dual to $s_{-k} = S_0$, which means S_{2k} is either X or Y. Indeed, in our example $s = S_5$, and $S_{2\times 5} = S_{10} = X$.

Similarly, if $s' = S_{-k}$ is the last T_4 state, we see that S_{-2k} is either X or Y, and s' is the only T_4 state between S_0 and S_{-2k}. Since X and Y are connected by a regular move, there are at most two T_4 states (s and s') in total.

The other case is where the graph of regular moves is two cycles. If there's any T_4 state on the cycle containing S_0, then the previous paragraphs show that that cycle contains X and Y too, so there's only one cycle, a contradiction. So all the T_4 states lie on the other cycle, that containing X. Let s be the first T_4 state in the sequence $X = X_0, X_1, \ldots, X_k = s$. Since $X = X_0 = s_{-k}$, by duality we have $X_{2k} = s_k = S_0$, which is dual to X. This means that X lies on the same cycle as S_0, a contradiction again. $\square$

5 Arbitrary starting state

This section shows that $d(s, s') \leq A + 3$ for any two states s and s'. If s' is reachable (from S_0), then $d(s, s') \leq d(s, S_0) + d(S_0, s') \leq 2 + (A + 1) = A + 3$. Note that $d(s, S_0) \leq 2$ because of the sequence $s = (a, b, c) \to (a + b, 0, c) \to (a + b + c, 0, 0) = S_0$.

This leaves us with the case of s' being unreachable. Consider the shortest path P from s to s'. None of the states in P are reachable, which means that all states in P are regular, that is, P does not contain any of S_{-1}, S_0, S_1, X, Y. This in turn implies that all moves in P are regular. Consider the unique sequence $s = s_0$, s_1, ... of regular moves. This sequence must contain s'. As explained in the proof of Theorem 4.1, this sequence is a cycle of length say m. This implies that $d(s, s') \leq m/2$. I'll show that $m \leq 2A + 1$, which implies the assertion.

The reasoning depends on whether the cycle s_0, s_1, ... contains any T_4 states. If no, then $m \leq 2A + 1$ because there only $2A + 1$ non T_4 states in total. This implies $d(s, s') \leq A$, as needed.

Let's say the cycle s_0, s_1, ... contains a T_4 state x. Consider the sequences

$x = x_0, x_1, x_2 \ldots$ and $x = x_0, x_{-1}, x_{-2}, \ldots$. As explained in the proof of Theorem 4.2, x_k and x_{-k} are dual for any k. Let x_k be the first T_4 state in x_1, x_2, ... Such x_k must existing because $x_0, x_1, \ldots$ is a cycle and x_0 is T_4. Since x_k is T_4, its dual state $x_{-k} = x_k$. Therefore the cycle $x_0, x_1, \ldots$ can have at most two T_4 states (x_0 and x_k). So the total number of states in $x_0, x_1, \ldots$ is at most $2A + 3$, which implies $d(s, s') \leq A + 1$.

6 Arbitrary amount of water

So far, we've been assuming that the total amount of water in all the three cups is A. We can relax this condition and assume $a + b + c = W$ for some W.

My guess is that most of the reasoning is the previous sections holds if we just replace A with W everywhere. So I conjecture that $d(s, s') \leq W + 3$ for any two states s and s'.

Note that we can assume that $W \leq (A+B+C)/2$. This is because pouring water from one cup to another is equivalent to pouring the remaining air from the second cup to the first. The total space occupied by air and water is $A + B + C$, so by taking the substance (air or water) that occupies less total space and naming it "water", we get $W \leq (A + B + C)/2$. This leads to the bound $d(s, s') \leq (A + B + C)/2 + 3$.