AN ALGORITHM TO COMPUTE THE EXACT SET OF INTEGERS THAT ARISE FROM A GIVEN n IN PROBLEM OF THE WEEK 1196

WITOLD JARNICKI (KRAKÓW)

We start with the candidate set A of all reachable elements and fix a potential new element d. We are investigating the following set.

$$B := \{f(d) : f \in A[X], f(0) > 0\}.$$

We want to check whether $0 \in B$. In order to do that, we consider the following set.

$$C := \{f(d)/d^n : f \in A[X], f(0) > 0, n \in \mathbb{N}, \deg f < n, d^n | f(d)\}.$$

The set C is finite and easily computable. Additionally, $0 \in C$ iff $0 \in B$.

Let's now move on to how C is computed. Observe that the following holds.

$$C = \bigcup_{n=1}^{\infty} C_n,$$

where

$$C_n := \{f(d)/d^n : f \in A[X], f(0) > 0, \deg f < n, d^n | f(d)\}.$$

Obviously, we have the following.

$$C_1 = \{a/d : a \in A, a > 0, d|a\}.$$

Additionally, for each $n \in \mathbb{N}$, $n > 0$ we have the following.

$$C_{n+1} = \{(a + b)/d : a \in C_n, b \in A, d|a + b\}.$$

So, all that happens is that the program computes C_0 and keeps applying the inductive formula until two successive generations are equal. More precisely, we stop if the following holds for some $k \in \mathbb{N}$.

$$\bigcup_{n=1}^{k} C_n = \bigcup_{n=1}^{k+1} C_n.$$

At such a point no more computation is necessary and we can check whether 0 was reached.