POW 1188 from Stan Wagon "A Delicate Balancing Act"

Solution by Michael Elgersma August 22, 2014 draft 4

Given a cylindrically shaped can, partially filled with water, what is the water height before tipping, that allows the can of water to be tilted as far as possible without tipping all the way over?

This solution below considers the case where the water level of the tilted can does not intersect the can's top or bottom surface. The coordinate system has the origin at the center of the bottom of the can, the z axis goes up along the can's axis of rotational symmetry, and the balance point at the can's bottom rim has $y = 0$. The following parameters are given:

$$\rho: \quad water\ density$$

$$H: \quad can\ height$$

$$R: \quad can\ radius$$

$$m_1: \quad mass\ of\ empty\ can$$

The following variables are used:

h: $height\ of\ cylindrical\ subset\ of\ the\ water, in\ the\ tilted\ can$

m_2: $mass\ of\ water\ in\ cylindrical\ shape, that\ does\ not\ intersect\ can\ top\ or\ bottom$

m_3: $mass\ of\ water\ above\ m_2$

c_i: $center\ of\ gravity\ of\ mass\ m_i$

θ: $maximum\ tilt\ angle\ of\ can, that\ still\ allows\ it\ to\ balance\ on\ its\ outer\ rim$

φ: $angle\ about\ the\ rotational\ symmetry\ axis\ of\ can$

The mass and center of gravity of the can are:

$$m_1 \qquad c_1 = \frac{H}{2}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

The mass and center of gravity of the cylindrical portion of water are:

$$m_2 = \rho\pi R^2\, h \qquad c_2 = \frac{h}{2}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

The mass and center of gravity of the water above m_2 are:

$$m_3 = \rho \int_0^R \int_0^{2\pi} \big(R + r cos(\varphi)\big) tan(\theta)\, r\, d\varphi\, dr = \rho\pi R^3\, tan(\theta)$$

$$c_3 = \frac{\rho}{m_3} \int_0^R \int_0^{2\pi} \begin{bmatrix} r\, cos(\varphi) \\ r\, sin(\varphi) \\ h + \big(R + r\, cos(\varphi)\big)\dfrac{tan(\theta)}{2} \end{bmatrix} \big(R + r\, cos(\varphi)\big)\, tan(\theta)\, r\, d\varphi\, dr$$
$$= \begin{bmatrix} R/4 \\ 0 \\ h + \dfrac{5}{8} R\, tan(\theta) \end{bmatrix}$$

The center of gravity of the combined can and water is:

$$c = \frac{m_1 c_1 + m_2 c_2 + m_3 c_3}{m_1 + m_2 + m_3} = \frac{m_1 \dfrac{H}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \rho\pi R^2\, h\, \dfrac{h}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \rho\pi R^3\, tan(\theta) \begin{bmatrix} R/4 \\ 0 \\ h + \dfrac{5}{8} R\, tan(\theta) \end{bmatrix}}{m_1 + \rho\pi R^2\, h + \rho\pi R^3\, tan(\theta)}$$

The can is balanced on its rim at point $\begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$ when the vertical vector from c to $\begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$ is orthogonal to the horizontal vector $\begin{bmatrix} cos(\theta) \\ 0 \\ sin(\theta) \end{bmatrix}$. This happens when $R\, cos(\theta) = c^T \begin{bmatrix} cos(\theta) \\ 0 \\ sin(\theta) \end{bmatrix}$, i.e. when:

$$R\, cos(\theta)$$
$$= \frac{m_1 \dfrac{H}{2} sin(\theta) + \rho\pi R^2\, h\, \dfrac{h}{2} sin(\theta) + \rho\pi R^3\, tan(\theta) \left[\dfrac{cos(\theta) R}{4} + sin(\theta) \left(h + \dfrac{5}{8} R\, tan(\theta) \right) \right]}{m_1 + \rho\pi R^2\, h + \rho\pi R^3\, tan(\theta)}$$

Clearing denominators and rearranging gives a quadratic polynomial in h:

$$d_2 h^2 + d_1 h + d_0 = 0$$

where

$$d_2 = \rho\pi R^2\, sin(\theta)/2$$

$$d_1 = \rho\pi R^3 \big(tan(\theta)\, sin(\theta) - cos(\theta) \big)$$

$$d_0 = m_1 \frac{H}{2} sin(\theta) + \rho\pi \frac{R^4}{4}\ tan(\theta) \left(cos(\theta) + \frac{5}{2} tan(\theta) sin(\theta) \right) - (m_1 + \rho\pi R^3\ tan(\theta)\ )\ R\ cos(\theta)$$

The solution of the quadratic equation for h is given by:

$$h = \frac{-d_1 \pm \sqrt{d_1 d_1 - 4 d_2 d_0}}{2 d_2}$$

Increasing the can tilt angle θ, until the solutions for h just stops being real, occurs when the radical is 0:

$$d_1 d_1 - 4 d_2 d_0 = 0$$

Expanding this equation gives:

$$0 = d_1 d_1 - 4 d_2 d_0 = \frac{\rho\pi R^2\ f}{4\left(cos(\theta)\right)^2}$$

where f is a homogeneous degree-4 polynomial in $cos(\theta)$ and $sin(\theta)$:

$$f = 4\pi\rho R^4 \left(cos(\theta)\right)^4 + 8\ m_1\ R \left(cos(\theta)\right)^3 sin(\theta) - (4 m_1\ H + 2\pi\rho R^4)\left(cos(\theta)\right)^2 \left(sin(\theta)\right)^2 - \pi\rho R^4 \left(sin(\theta)\right)^4$$

A second polynomial equation in $cos(\theta)$ and $sin(\theta)$ is given by:

$$g = \left(cos(\theta)\right)^2 + \left(sin(\theta)\right)^2 - 1 = 0$$

Solving simultaneous equations $f = 0$ and $g = 0$, gives 8 solutions for $[cos(\theta), sin(\theta)]$.

The 8 solutions come in 4 pairs, since replacing $[cos(\theta), sin(\theta)]$ with $[-cos(\theta), -sin(\theta)]$ leaves equations $f = 0$ and $g = 0$ unchanged. After solving for $[cos(\theta), sin(\theta)]$, those values can be put into the equation for h, with the radical being zero:

$$h = \frac{-d_1}{2 d_2}$$

The height of water in the can, after it is tilted back to level, is given by:

$$h_{level} = \frac{m_2 + m_3}{\rho\pi R^2} = h + R\ tan(\theta)$$

Example: When $H = 13\ cm$, $R = 7/2\ cm$, $m_1\ = 50\ grams$ and $\rho = 1\ gram/(cm)^3$ then Mathematica gives eight solutions to Solve[f==0 && g==0, {cosTheta,sinTheta}]. The only solution that is both real and lies in the range $0 < \theta < \pi/2$ is:

$$cos(\theta) = 0.7452426811917$$

$$sin(\theta) = 0.6667933308981$$

Using these values of $[cos(\theta), sin(\theta)]$ to evaluate $[d_2, d_1, d_0]$ then h and h_{level}, gives:

$$h_{level} = 3.9117808521836 \text{ cm}$$

Although only $[cos(\theta), sin(\theta)]$ and their ratio, are needed in the computation of h_{level}, the associated value of θ can be computed to be:

$$\theta = 0.7298976070438\ rad =\ 41.8200523603119\ degrees$$