Problem 1182. A Power Sum Puzzle from Stan Wagon June 2014

Suppose $x_1, x_2, \cdots, x_n$ are n complex numbers and $b_1, b_2, \cdots, b_n$ are real numbers such that

$$x_1 + x_2 + \cdots + x_n = b_1$$

$$(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2 = b_2$$

$$\vdots$$

$$(x_1)^n + (x_2)^n + \cdots + (x_n)^n = b_n$$

What is the value of $(x_1)^m + (x_2)^m + \cdots + (x_n)^m$, i.e. b_m, for integer powers $m > n$?

Source: Due to Michael Elgersma, Stan Wagon, extending an old chestnut involving three variables and 3, 5, 7 on the right side.

If $n \times n$ matrix A has eigenvalues $x_1, x_2, \cdots, x_n$ then the characteristic equation of A is:

$$0 = det(\lambda I - A) = \prod_{k=1}^{n} (\lambda - x_k) = \lambda^n - \sum_{k=1}^{n} c_k \lambda^{n-k}$$

where

$$c_k = (-1)^{k+1} \left(\sum_{1=i1<i2<\cdots<ik}^{n} x_{i1}\, x_{i2} \cdots\, x_{ik} \right)$$

The coefficients c_k can be solved for in terms of coefficients b_i using Newton's identities [S]:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ b_1 & 2 & 0 & \cdots & 0 \\ b_2 & b_1 & 3 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ b_{n-1} & \cdots & b_2 & b_1 & n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$c_1 = b_1 \qquad c_2 = \tfrac{1}{2}[b_2 - (b_1)^2] \qquad c_3 = \tfrac{1}{6}[2b_3 - 3b_1 b_2 + (b_1)^3]$$

$$c_4 = \tfrac{1}{24}[6b_4 - 3(b_2)^2 - 8b_3 b_1 + 6(b_1)^2 b_2 - (b_1)^4] \qquad \ldots$$

The characteristic equation is used again to get b_m, for integer powers $m > n$, from the c_k coefficients. Since $0 = det(AI - A)$, a matrix satisfies its own characteristic equation $A^n = \sum_{k=1}^{n} c_k A^{n-k}$. Multiplying this last equation by A^m gives:

$$A^{n+m} = \sum_{k=1}^{n} c_k A^{n-k+m}$$

Since $trace(A^k) = (x_1)^k + (x_2)^k + \cdots + (x_n)^k = b_k,$ taking trace of the last equation gives a recursion formula for the b_i coefficients:

$$b_{n+m} = \sum_{k=1}^{n} c_k b_{n+m-k}$$

Example: Let $n = 4$ and let the b_i coefficients be: $b_1 = 1,\ b_2 = 0,\ b_3 = 1,\ b_4 = 0.$

The coefficient formulas derived on the previous page then give:

$$c_1 = b_1 = 1 \qquad c_2 = \tfrac{1}{2}[b_2 - (b_1)^2] = -1/2 \qquad c_3 = \tfrac{1}{6}[2b_3 - 3b_1b_2 + (b_1)^3] = 1/2$$

$$c_4 = \tfrac{1}{24}[6b_4 - 3(b_2)^2 - 8b_3b_1 + 6(b_1)^2b_2 - (b_1)^4] = -3/8$$

Using these values in the recursion formula for the b_i coefficients:

$$b_5 = \sum_{k=1}^{4} c_k b_{4+1-k} = c_1b_4 + c_2b_3 + c_3b_2 + c_4b_1 = 0 - \frac{1}{2} + 0 - \frac{3}{8} = -\frac{7}{8}$$

$$b_6 = \sum_{k=1}^{4} c_k b_{4+2-k} = c_1b_5 + c_2b_4 + c_3b_3 + c_4b_2 = -\frac{7}{8} + 0 + \frac{1}{2} + 0 = -\frac{3}{8}$$

$$b_7 = \sum_{k=1}^{4} c_k b_{4+3-k} = c_1b_6 + c_2b_5 + c_3b_4 + c_4b_3 = -\frac{3}{8} + \left(\frac{1}{2}\right)\frac{7}{8} + 0 - \frac{3}{8} = -\frac{5}{16}$$

$$b_8 = \sum_{k=1}^{4} c_k b_{4+4-k} = c_1b_7 + c_2b_6 + c_3b_5 + c_4b_4 = -\frac{5}{16} + \left(\frac{1}{2}\right)\frac{3}{8} - \left(\frac{1}{2}\right)\frac{7}{8} + 0 = -\frac{9}{16}$$

$$b_9 = \sum_{k=1}^{4} c_k b_{4+5-k} = c_1b_8 + c_2b_7 + c_3b_6 + c_4b_5 = -\frac{9}{16} + \left(\frac{1}{2}\right)\frac{5}{16} - \left(\frac{1}{2}\right)\frac{3}{8} + \left(\frac{3}{8}\right)\frac{7}{8} = -\frac{17}{64}$$

$$b_{10} = \sum_{k=1}^{4} c_k b_{4+6-k} = c_1b_9 + c_2b_8 + c_3b_7 + c_4b_6 = -\frac{17}{64} + \left(\frac{1}{2}\right)\frac{9}{16} - \left(\frac{1}{2}\right)\frac{5}{16} + \left(\frac{3}{8}\right)\frac{3}{8} = 0$$

Reference: [S] Raymond Seroul, “Programming for Mathematicians,” Springer, 2000.